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Construction of a matrix product stationary state from solutions of a finite-size system

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Abstract
Stationary states of stochastic models, which have N states per site, in matrix product form are considered. First we give a necessary condition for the existence of a finite M-dimensional matrix product stationary state for any $\{N, M\}$. Second, we give a method to construct the matrices from the stationary states of small-size systems when the above condition and $N \leqslant M$ are satisfied. Third, the method by which one can check that the obtained matrices are valid for any system size is presented for the case where $M = N$ is satisfied. The application of our methods is explained using three examples: the asymmetric simple exclusion process, a model studied by Jafarpour (2003 *J. Phys. A: Math. Gen.* **36** 7497) and a hybrid of both models.

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1. Introduction

One-dimensional many-body stochastic models have attracted much attention because of their rich nonequilibrium behaviours and wide applicability in condensed matter physics, biology and other fields [1–4]. The models also have fundamental aspects: they serve as models of which we can inspect the behaviours and, through them, develop theories of nonequilibrium statistical physics.

To study the properties of such models, there are several methods. One might start from performing Monte Carlo simulations to get a rough grasp about what is going on. Then one might apply some approximate methods such as mean field theory or the cluster approximation. One might also adopt renormalization group techniques. In many cases these kinds of analyses give us satisfactory understanding of the models.

In some cases, however, these methods are not enough to fully understand the properties of the models. Monte Carlo results and approximations sometimes disguise the physics of the models. For instance in [5] it was shown that a numerically apparent phase transition associated with a particle condensation is not accompanied by a non-analyticity of physical quantities. In such cases, it would be desirable to have exact solutions of the model. Of course most models do not admit exact analytical treatment. But for some models, particularly for those in one dimension, one can obtain exact solutions and study the properties of the models in some detail.

In fact there is a class of models for which the exact stationary state can be written in a matrix product form [6]. We abbreviate this matrix-product stationary state as MPSS in the following. After the discovery of an exact matrix-product groundstate for a one-dimensional quantum spin chain [7], the first exact MPSS of the one-dimensional stochastic models was found in [8] for the asymmetric simple exclusion process (ASEP) (section 3.1). It is interesting that numerical solutions in the matrix product form (MPF) also play an important role [9, 10] in the method of density matrix renormalization group (DMRG) [11] and its higher-dimensional extension [12, 13] through a generalization of MPF (tensor product form).

Following [8], the MPSSs have been found for many one-dimensional stochastic models. In addition, it is known that stationary states of stochastic Hamiltonians

- with nearest-neighbour interaction (this is the interaction in the current paper, treated in (2.46)–(2.49)) or
- with arbitrary, but finite, interaction range

can be expressed as *infinite*-dimensional MPSSs (see [14] and [15], respectively). So far, however, it has not been known how to construct systematically the *finite*-dimensional MPSS for a model at hand. One might try very hard to find even a two-dimensional matrix product state for several days, in vain. Hence, it would be desirable to have some methods to know

1. whether the stationary state of the model at hand can be written in a MPF or not,
2. if yes, how to find the matrices in the MPF.

The objective of this paper is to partially answer this question. More precisely, for the case where the dimensions (M) of the matrices are finite, we give

1. for any $\{N, M\}$, a way to find necessary conditions for the existence of an exact M-dimensional MPSS of stochastic models which have N states per site;
2. a systematic way by which the exact concrete M-dimensional representation of the MPF can be constructed from the stationary states of a small size system if the condition obtained in the above-mentioned step 1 is also a sufficient condition and $N \leqslant M$ is satisfied;
3. for the case $M = N$, the method by which one can check that the obtained matrices are valid for any system size.

This paper is organized as follows: in section 2, we describe our method. Three examples are given in section 3. Section 4 is devoted to the summary and a list of possible extensions of our methods.

Hereafter, we omit the word 'exact'; namely, we use 'a MPSS' and 'a representation' instead of 'an exact MPSS' and 'an exact representation', respectively.

2. Method

Let us consider a many-particle stochastic model on a chain of size L. Each site j ($j = 1, 2, \ldots, L$) is assumed to be in one of N states which we denote as τ_j $(= 0, 1, 2, \ldots, (N-1))$.

For instance when $N = 2$, $\tau_j = 0$ (resp. $\tau_j = 1$) may represent a state in which the site j is empty (resp. occupied by a particle). In this paper we only consider a model in a continuous time setting, for which the dynamics can be described by the master equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}_L(t) = -H\vec{P}_L(t). \tag{2.1}$$

Here $\vec{P}_L(t)$ is a vector whose component represents the probability of a system in a certain state at time t. H is a transition rate matrix describing the stochastic dynamics of the model. This formulation of a stochastic model is called a quantum Hamiltonian formalism of the master equation because of the obvious similarity of (2.1) to the imaginary-time Schrödinger equation [16]. In a stationary state, to which we restrict our attention in the following, the left-hand side of (2.1) vanishes. The stationary state vector, $\vec{P}_L$, of the model (size L) is determined by

$$H\vec{P}_L = 0. \tag{2.2}$$

In this paper, we are interested in a very special situation in which $\vec{P}_L$ can be written in a matrix product form (MPF),

$$\vec{P}_L = \vec{P}_L^{\mathrm{MPF}} \qquad \text{for} \quad L = 1, 2, 3, \ldots, \tag{2.3}$$

where $\vec{P}_L^{\mathrm{MPF}}$ is the vector whose component $P_L^{\mathrm{MPF}}(\tau_1, \tau_2, \ldots, \tau_L)$ is defined by

$$P_L^{\mathrm{MPF}}(\tau_1, \tau_2, \ldots, \tau_L) := \frac{1}{Z_L}\langle W|A(\tau_1)A(\tau_2)\cdots A(\tau_L)|V\rangle. \tag{2.4}$$

Here $\{A(\tau)\}_{\tau=0,1,\ldots,(N-1)}$ are M-dimensional square matrices with M being assumed to be finite. $\langle W|$ (resp. $|V\rangle$) is an M-dimensional row (resp. column) vector. Z_L is the normalization constant defined by

$$Z_L := \langle W|[A(0) + A(1) + \cdots A(N-1)]^L|V\rangle. \tag{2.5}$$

We will refer to $\{A(0), \ldots, A(N-1), \langle W|, |V\rangle\}$ as 'a set of matrices' for simplicity in the sequel. One should notice that there is a trivial freedom of a similarity transformation for the choice of the matrices in (2.4). If one introduces another set of matrices $\{\widetilde{A}(0), \ldots, \widetilde{A}(N-1), \langle\widetilde{W}|, |\widetilde{V}\rangle\}$ by

$$\begin{aligned} &\langle W|S =: \langle\widetilde{W}|, \\ &S^{-1}A(\tau)S =: \widetilde{A}(\tau), \qquad (\tau = 0, 1, \ldots, N-1) \\ &S^{-1}|V\rangle =: |\widetilde{V}\rangle, \end{aligned} \tag{2.6}$$

one has

$$P_L^{\mathrm{MPF}}(\tau_1, \tau_2, \ldots, \tau_L) = \frac{1}{Z_L}\langle\widetilde{W}|\widetilde{A}(\tau_1)\widetilde{A}(\tau_2)\cdots\widetilde{A}(\tau_L)|\widetilde{V}\rangle. \tag{2.7}$$

It is known that, $\vec{P}_L$ of some models has the MPF when some conditions of the model parameters are satisfied.

In the following discussions, we sometimes explain the main ideas using the $N = 2$ case, in which case two matrices $A(0)$, $A(1)$ are renamed as $A(0) =: E$, $A(1) =: D$. Then the MPF may be written as

$$\vec{P}_L = \frac{1}{Z_L}\langle W|\begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L}|V\rangle = \frac{1}{Z_L}\langle W|\underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \begin{pmatrix} E \\ D \end{pmatrix}}_{L \quad \text{times}}|V\rangle. \tag{2.8}$$

For later use, we also define

$$P^{m,n} := \frac{1}{Z_{m+n}} \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes m} (E\ D)^{\otimes n} |V\rangle, \tag{2.9}$$

which is a conversion of the vector $\vec{P}_L$ to a matrix form with $L = m + n$. Notice that the rank of this matrix is at most $2^{\min\{m,n\}}$, where $\min\{m, n\}$ is a function whose value is the smaller number of m and n.

2.1. How to find necessary conditions for the existence of an MPSS

In this subsection, we give a way to find necessary conditions for the existence of an M-dimensional MPSS. The key observation here is that when the $\vec{P}_L$ has an M-dimensional MPF, the rank of a matrix $P^{m,n}$ obtained from $\vec{P}_L$ is no larger than M. Hence, if we take m and n satisfying $N^m > M$, $N^n > M$, the rank of the matrix $P^{m,n}$ is M which is strictly smaller than $N^{\min\{m,n\}}$. We call this phenomenon the 'rank deficiency'. The rank deficiency occurs when the $\vec{P}_L$ has an M-dimensional MPF. In reverse, it is by this rank deficiency that we can find necessary conditions for the existence of an M-dimensional MPSS.

First we explain this using the case where $N = M = 2, L = 4$. Let us consider equation (2.2) for $L = 4$ and suppose that the solution $\vec{P}_4$ has the MPF,

$$\vec{P}_4 = \frac{1}{Z_4} \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 4} |V\rangle. \tag{2.10}$$

We convert the vector form (2.10) into the matrix form,

$$P^{2,2} = \frac{1}{Z_4} \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} (E\ D)^{\otimes 2} |V\rangle, \tag{2.11}$$

which is a 4×4 matrix.

Now we show that, when $M = 2$, the rank of $P^{2,2}$ is at most two, i.e., the rank deficiency of $P^{2,2}$ occurs. Since $\{EE|V\rangle, ED|V\rangle, DE|V\rangle, DD|V\rangle\}$ is a set of $M(= 2)$-dimensional vectors, we can prepare two basis (column) vectors, which we call $|e_1\rangle, |e_2\rangle$. There exist some coefficients, $\{a_k, b_k\}_{k=1,2,3,4}$, such that

$$EE|V\rangle = a_1|e_1\rangle + b_1|e_2\rangle, \tag{2.12}$$

$$ED|V\rangle = a_2|e_1\rangle + b_2|e_2\rangle, \tag{2.13}$$

$$DE|V\rangle = a_3|e_1\rangle + b_3|e_2\rangle, \tag{2.14}$$

$$DD|V\rangle = a_4|e_1\rangle + b_4|e_2\rangle. \tag{2.15}$$

Using these, column vectors in the RHS of (2.11) are

$$\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} EE|V\rangle = a_1 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_1\rangle + b_1 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_2\rangle, \tag{2.16}$$

$$\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} ED|V\rangle = a_2 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_1\rangle + b_2 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_2\rangle, \tag{2.17}$$

$$\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} DE|V\rangle = a_3 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_1\rangle + b_3 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_2\rangle, \tag{2.18}$$

$$\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} DD|V\rangle = a_4 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_1\rangle + b_4 \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_2\rangle. \tag{2.19}$$

One notices that the right-hand sides of (2.16)–(2.19) are written as linear combinations of the two vectors,

$$\left\{ \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |e_i\rangle \right\}_{i=1,2}. \tag{2.20}$$

Hence there are at most two independent vectors among the four column vectors of $P^{2,2}$. This means that the rank of $P^{2,2}$ is at most two. The generalization of the above argument to general M, N, L case is not difficult.

One should notice that this can be used for checking the existence of a finite-dimensional MPSS for a given model. For instance to be sure that there is no ($M =$) 2-dimensional MPSS for a given parameter set of an $N = 2$ model, all one needs to do is to find a stationary state for $L = 4$ and see that the rank deficiency does not occur.

Furthermore we can use this rank deficiency to find necessary conditions for the existence of an M-dimensional MPSS. Algorithmically one performs the following steps.

1. Solve equation (2.2) for L for which one can take m, n such that $L = m + n$ and $N^m > M, N^n > M$.
2. Make the $N^m \times N^n$ matrix $P^{m,n}$ from $\vec{P}_L$.
3. Calculate the rank of the matrix $P^{m,n}$.
4. Find necessary conditions for the existence of an M-dimensional MPSS, from conditions that the rank of $P^{m,n}$ is M.

For instance, for the case where $M = N = 2, L = 4$, one finds P_4, converts it to $P^{2,2}$, computes the rank of it with the model parameters unfixed and looks for a condition where the rank deficiency occurs. If one finds a condition where the rank of $P^{2,2}$ is 2, it is a necessary condition for the existence of an $M(= 2)$-dimensional MPSS.

Note that a necessary condition found in step 4 may not be a sufficient condition for the existence of a finite-dimensional MPSS. In fact there occurs a different type of rank deficiency, caused by the existence of algebraic relations. This really happens for the ASEP and will be explained at the end of section 3.1.1.

In practice it is in general not an easy task to calculate the rank of a matrix with a lot of parameters unfixed. But if we use a computer algebra system such as Maple (we have used Maple to perform most of the calculations in section 3), it is possible to do this to some extent. There are several methods to compute the rank of a matrix on a computer. When explaining our methods in the next section, we will adopt a version of Gaussian elimination procedure which consists of several elementary transformations. By using elementary transformations, which do not change the rank of a matrix, we transform $P^{m,n}$ into a matrix as diagonal as possible so that the resultant matrix takes the form,

$$\left[\begin{array}{c|c} I_r & B \\ \hline \multicolumn{2}{c}{O} \end{array} \right], \tag{2.21}$$

where I_r, O and B represents the $r \times r$ identity matrix, an $(N^m - r) \times N^n$ zero matrix and an $r \times (N^n - r)$ matrix, respectively. We can tell that the rank of (2.21) is r. Thus the procedure for finding necessary conditions for the existence of an M-dimensional MPSS is rewritten as follows.

1. Solve equation (2.2) for L for which one can take m, n such that $L = m + n$ and $N^m > M, N^n > M$.
2. Make the $N^m \times N^n$ matrix $P^{m,n}$ from $\vec{P}_L$.
3. Perform a set of the elementary transformations of $P^{m,n}$ so that the resultant matrix has the form of (2.21).

4. Find necessary conditions for the existence of an *M*-dimensional MPSS, from conditions that the rank of the resultant matrix is *M*.

2.2. How to construct the matrix-product stationary state

Suppose that a necessary condition obtained by the method in section 2.1 is also a sufficient condition for the existence of an *M*-dimensional MPSS. Then, how can we find the set of matrices $\{A(0), A(1), \ldots, A(N-1), \langle W|, |V\rangle\}$ in P_L^{MPF} (equation (2.4)) from $\vec{P}_L$ for a finite number of *L*? We consider this problem in section 2.2.1. In the subsequent section 2.2.2, we comment on a way to check whether the obtained matrices can be used for constructing $\vec{P}_L$ for any system size *L* for the case $N = M$.

2.2.1. Finding the matrices. Again we first explain the idea for the case where $N = M = 2$. The first thing to do is to solve (2.2) for $L = 2, 3$ and get $\vec{P}_{L=2}$ and $\vec{P}_{L=3}$. If the stationary state $\vec{P}_L$ can be written in the form as (2.4), one has

$$Z_2\vec{P}_{L=2} = \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 2} |V\rangle, \tag{2.22}$$

$$Z_3\vec{P}_{L=3} = \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes 3} |V\rangle. \tag{2.23}$$

The matrix form $P^{1,1}$ of (2.22) is

$$Z_2 P^{1,1} := \langle W| \begin{pmatrix} E \\ D \end{pmatrix} (E\ D)|V\rangle. \tag{2.24}$$

Let us recall the freedom of the choice of matrices in (2.6) and (2.7) by a similarity transformation. If we introduce

$$\begin{aligned} \langle W|S &=: \langle \widetilde{W}|, \\ S^{-1}ES &=: \widetilde{E}, \\ S^{-1}DS &=: \widetilde{D}, \\ S^{-1}|V\rangle &=: |\widetilde{V}\rangle, \end{aligned} \tag{2.25}$$

with *S* an invertible 2×2 matrix, $Z_2P^{1,1}$ in (2.24) can also be expressed as

$$Z_2 P^{1,1} = \langle \widetilde{W}| \begin{pmatrix} \widetilde{E} \\ \widetilde{D} \end{pmatrix} (\widetilde{E}\ \widetilde{D})|\widetilde{V}\rangle. \tag{2.26}$$

Similarly for $L = 3$, one has

$$Z_3 P^{1,2} = \langle \widetilde{W}| \begin{pmatrix} \widetilde{E} \\ \widetilde{D} \end{pmatrix} (\widetilde{E}\ \widetilde{D})^{\otimes 2}|\widetilde{V}\rangle. \tag{2.27}$$

The key of our methods is to choose *S* in (2.25)–(2.27) to be

$$S := (E|V\rangle\ D|V\rangle). \tag{2.28}$$

In this equation, both $E|V\rangle$ and $D|V\rangle$ are two-dimensional column vectors, so that this *S* is a 2×2 matrix. For this special choice of *S*, the following holds:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (\widetilde{E}|\widetilde{V}\rangle\ \widetilde{D}|\widetilde{V}\rangle). \tag{2.29}$$

Using this equation (2.29) and (2.26), one has

$$Z_2 P^{1,1} = \begin{pmatrix} \langle\widetilde{W}|\widetilde{E} \\ \langle\widetilde{W}|\widetilde{D} \end{pmatrix}. \tag{2.30}$$

It also follows from (2.25) that

$$(E\ D)S = S(\widetilde{E}\ \widetilde{D}). \tag{2.31}$$

By virtue of this S one has

$$\begin{aligned}(E\ D) \otimes (E\ D)|V\rangle &= (E(E\ D)\ D(E\ D))|V\rangle \\ &= (E(E|V\rangle\ D|V\rangle)\ D(E|V\rangle\ D|V\rangle)) \\ &= (ES\ DS)(\because \text{equation (2.28)}) \\ &= S(\widetilde{E}\ \widetilde{D})(\because \text{equation (2.31)}).\end{aligned} \tag{2.32}$$

Namely,

$$(E\ D)^{\otimes 2}|V\rangle = (E\ D)|V\rangle(\widetilde{E}\ \widetilde{D}). \tag{2.33}$$

From this equation (2.33), we can obtain

$$\langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes k} (E\ D)^{\otimes(j+1)}|V\rangle = \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes k} (E\ D)^{\otimes j}|V\rangle(\widetilde{E}\ \widetilde{D}) \tag{2.34}$$

for $k, j = 1, 2, 3, \ldots$.

An application of (2.34) for $k = j = 1$ leads to

$$Z_3 P^{1,2} = Z_2 P^{1,1}(\widetilde{E}\ \widetilde{D}), \tag{2.35}$$

from which one obtains

$$\widetilde{E} = (Z_2 P^{1,1})^{-1}(Z_3 P^{1,2}[1:2, 1:2]), \tag{2.36}$$

$$\widetilde{D} = (Z_2 P^{1,1})^{-1}(Z_3 P^{1,2}[1:2, 3:4]). \tag{2.37}$$

Here, we introduced the notation $A[b : c, d : e]$ for a submatrix of a matrix A constructed by selecting the row range from the bth row to the cth row and the column range from the d-th column to the e-th column. Now remember that $Z_3 P^{1,2}$ and $Z_2 P^{1,1}$ of these equations can be obtained by solving equation (2.2) for a small size $L = 2, 3$. Therefore we can obtain $(\widetilde{E}\ \widetilde{D})$ from the solutions for $L = 2, 3$.

It should be noted that we do not need an explicit expression of S in calculating $\{\widetilde{E}, \widetilde{D}\}$ (equations (2.36) and (2.37)). S appears only in the equations from which equations (2.36) and (2.37) are derived.

The case $2 = N < M$ is treated as follows. Instead of (2.28), we define the $M \times M$ matrix S as

$$S := (E\ D)^{\otimes\ell}|V\rangle U, \tag{2.38}$$

where ℓ is an integer which satisfies $2^\ell \geqslant M$. U is a $2^\ell \times M$ matrix by which we choose mutually independent M column vectors among 2^ℓ column vectors of a matrix $(E\ D)^{\otimes\ell}|V\rangle$ so that there exists S^{-1}. It is noted that when $M = 2^\ell$ ($\ell = 2, 3, 4, \ldots$) and $M (= 2^\ell)$ column vectors of a matrix $(E\ D)^{\otimes\ell}|V\rangle$ are mutually independent, we can set U as the $M \times M$ identity matrix.

It sometimes really happens that 2^ℓ column vectors of a matrix $(E\ D)^{\otimes\ell}|V\rangle$ are NOT mutually independent. One such case is that of the four-dimensional set of matrices for the asymmetric simple exclusion process (ASEP) (section 3.1). In this case of $M = 2^2$ we cannot use equation (2.38) with $\ell = 2$ and $U =$ (the four-dimensional identity matrix). This is because only three in all four column vectors of $(E\ D)^{\otimes\ell}|V\rangle$ are mutually independent. This stems from (3.6) and the existence of the algebraic relation (3.5). For details, see the last paragraph of section 3.1.1.

We should note that equation (2.31) is also satisfied in this case. Hence, as a generalization of (2.33) one has

$$(E\ D)^{\otimes(j+1)}|V\rangle U = (E\ D)^{\otimes j}|V\rangle U(\widetilde{E}\ \widetilde{D}) \tag{2.39}$$

for $j = \ell, \ell+1, \ell+2, \ldots$. Therefore

$$T\langle W|\begin{pmatrix} E \\ D \end{pmatrix}^{\otimes k}(E\ D)^{\otimes(j+1)}|V\rangle U = T\langle W|\begin{pmatrix} E \\ D \end{pmatrix}^{\otimes k}(E\ D)^{\otimes j}|V\rangle U(\widetilde{E}\ \widetilde{D}), \tag{2.40}$$

that is,

$$TZ_{k+j+1}P^{k,j+1}U = TZ_{k+j}P^{k,j}U(\widetilde{E}\ \widetilde{D}) \tag{2.41}$$

for $j = \ell, \ell+1, \ell+2, \ldots$ and $k = 1, 2, 3, \ldots$. Here T is a $M \times 2^k$ matrix by which we choose mutually independent M row vectors among 2^k row vectors of a matrix $Z_{k+j}P^{k,j}U$ in the RHS of (2.41) so that we can solve for $\widetilde{E}$ and $\widetilde{D}$ in (2.41) as in (2.36) and (2.37). Equation (2.41) is the generalization of (2.35).

Generalization to the case $2 < N \leqslant M$ is straightforward. We define S by

$$S := (A(0)\ A(1) \cdots A(N-1))^{\otimes\ell}|V\rangle U, \tag{2.42}$$

where U is a $N^\ell \times M$ matrix. In this case, from (2.6), we have

$$I_M = (\widetilde{A}(0)\ \widetilde{A}(1) \cdots \widetilde{A}(N-1))^{\otimes\ell}|\widetilde{V}\rangle U, \tag{2.43}$$

where I_M is the M-dimensional identity matrix. This generalizes the equation (2.29). If $N = M$ and all column vectors in $(A(0)\ A(1) \cdots A(N-1))|V\rangle$ are mutually independent, then we can choose $\ell = 1$ and $U =$ (the identity matrix) in (2.42) and (2.43). Namely, they are simplified into the following equations:

$$S := (A(0)\ A(1) \cdots A(N-1))|V\rangle \tag{2.44}$$

and

$$I_N = (\widetilde{A}(0)\ \widetilde{A}(1) \cdots \widetilde{A}(N-1))|\widetilde{V}\rangle, \tag{2.45}$$

respectively.

2.2.2. Validity of the obtained matrices for arbitrary size L. Now that we have a set of matrices $\{\widetilde{A}(0), \widetilde{A}(1), \ldots, \widetilde{A}(N-1), |\widetilde{V}\rangle, \langle\widetilde{W}|\}$ by the method described in section 2.2.1, we would like to know whether the obtained set of matrices is valid for *any* system size L or not. It would not be difficult to check this for each small values of L using a computer, but this does not guarantee its validity for general L.

In some cases, however, one can check the validity of the obtained set of matrices for any L. In the following discussions we assume that the following two conditions are satisfied:

- $N = M$ and all column vectors in $(A(0)\ A(1) \cdots A(N-1))|V\rangle$ are mutually independent, where we can adopt equation (2.44) as S.
- The total Hamiltonian H has the form

$$H = h^{(\mathrm{L})} + \sum_{i=1}^{L-1} h_i + h^{(\mathrm{R})}. \tag{2.46}$$

Let I denote the N-dimensional identity matrix. Then each term in (2.46) takes the form,

$$h_i := I^{\otimes(i-1)} \otimes h_{\rm int} \otimes I^{\otimes(L-i-1)}, \tag{2.47}$$

$$h^{(\rm L)} := h^{(\ell)} \otimes I^{\otimes(L-1)}, \tag{2.48}$$

$$h^{(\rm R)} := I^{\otimes(L-1)} \otimes h^{(\rm r)}. \tag{2.49}$$

Now suppose that we have another set of matrices $\{A_{\rm c}(0), A_{\rm c}(1), \ldots, A_{\rm c}(N-1)\}$ which satisfies the following three equations:

$$h_{\rm int}[\widetilde{\mathbf{A}}^{\otimes 2}] = \mathbf{A}_{\rm c} \otimes \widetilde{\mathbf{A}} - \widetilde{\mathbf{A}} \otimes \mathbf{A}_{\rm c}, \tag{2.50}$$

$$\langle \widetilde{W}|[h^{(\ell)}\widetilde{\mathbf{A}}] = -\langle \widetilde{W}|\mathbf{A}_{\rm c}, \tag{2.51}$$

$$[h^{(\rm r)}\widetilde{\mathbf{A}}]|\widetilde{V}\rangle = \mathbf{A}_{\rm c}|\widetilde{V}\rangle, \tag{2.52}$$

where $\widetilde{\mathbf{A}}$ and $\mathbf{A}_{\rm c}$ are defined as

$$\widetilde{\mathbf{A}} := \begin{pmatrix} \widetilde{A}(0) \\ \widetilde{A}(1) \\ \vdots \\ \widetilde{A}(N-1) \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{\rm c} := \begin{pmatrix} A_{\rm c}(0) \\ A_{\rm c}(1) \\ \vdots \\ A_{\rm c}(N-1) \end{pmatrix}, \tag{2.53}$$

respectively. Then we can show that the MPF constructed from the set of matrices $\{\widetilde{A}(0), \widetilde{A}(1), \ldots, \widetilde{A}(N-1), |\widetilde{V}\rangle, \langle\widetilde{W}|\}$ according to (2.4) solves equation (2.2) for *any* size L. This is due to the so-called cancellation mechanism [17–19]; all terms in $H\vec{P}_L^{\rm MPF} = h^{(\rm L)}\vec{P}_L^{\rm MPF} + \sum_{i=1}^{L-1} h_i \vec{P}_L^{\rm MPF} + h^{(\rm R)}\vec{P}_L^{\rm MPF}$ are cancelled out by (2.50)–(2.52).

A good property of our set of matrices $\{\widetilde{A}(0), \widetilde{A}(1), \ldots, \widetilde{A}(N-1), \langle\widetilde{W}|, |\widetilde{V}\rangle\}$ enables us to compute easily a candidate for $\{A_{\rm c}(0), A_{\rm c}(1), \ldots, A_{\rm c}(N-1)\}$ in the case $N = M$. Namely, we can calculate $\{A_{\rm c}(0), A_{\rm c}(1), \ldots, A_{\rm c}(N-1)\}$ from the formula

$$\mathbf{A}_{\rm c} = \begin{pmatrix} \widetilde{A}(0)\, {}^t h^{(\rm r)} \\ \widetilde{A}(1)\, {}^t h^{(\rm r)} \\ \vdots \\ \widetilde{A}(N-1)\, {}^t h^{(\rm r)} \end{pmatrix} + \Xi\{h_{\rm int}[\widetilde{\mathbf{A}}^{\otimes 2}]\}|\widetilde{V}\rangle, \tag{2.54}$$

where we denote a transpose of a matrix A, by ${}^tA$ and $\Xi\{\cdot\}$ is the operator which transforms an N^2-dimensional vector ${}^t(B_1, B_2, \ldots, B_{N^2}) =: \mathbf{B}[N^2]$ into

$$\begin{bmatrix} B_1 & B_2 & \cdots & B_N \\ B_{N+1} & B_{N+2} & \cdots & B_{2N} \\ B_{2N+1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & B_{N^2} \end{bmatrix}.$$

This operator satisfies $\Xi\{\mathbf{B}[N^2] + \mathbf{C}[N^2]\} = \Xi\{\mathbf{B}[N^2]\} + \Xi\{\mathbf{C}[N^2]\}$ and $\Xi\{\mathbf{B}[N] \otimes \mathbf{C}[N]\} = \mathbf{B}[N] \otimes {}^t\mathbf{C}[N]$. It is noted that the argument of the LHS of the latter is a vector with N^2 components.

Derivation of (2.54) is as follows. By virtue of (2.45) (the good property of our set of matrices), we can derive from (2.52)

$${}^th^{(\rm r)} = {}^t\mathbf{A}_{\rm c}|\widetilde{V}\rangle. \tag{2.55}$$

We perform $\Xi\{\text{equation (2.50)}\}|\widetilde{V}\rangle$ to get

$$\Xi\{h_{\text{int}}[\widetilde{\mathbf{A}} \otimes \widetilde{\mathbf{A}}]\}|\widetilde{V}\rangle = [\mathbf{A}_{\text{c}} \otimes {}^{t}\widetilde{\mathbf{A}}]|\widetilde{V}\rangle - [\widetilde{\mathbf{A}} \otimes {}^{t}\mathbf{A}_{\text{c}}]|\widetilde{V}\rangle. \tag{2.56}$$

Using (2.45), (2.55) and a relation $(A \otimes B)|\widetilde{V}\rangle = A \otimes (B|\widetilde{V}\rangle)$, the RHS of (2.56) is

$$\mathbf{A}_{\text{c}} \otimes [{}^{t}\widetilde{\mathbf{A}}|\widetilde{V}\rangle] - \widetilde{\mathbf{A}} \otimes [{}^{t}\mathbf{A}_{\text{c}}|\widetilde{V}\rangle] = \mathbf{A}_{\text{c}} \otimes I_N - \widetilde{\mathbf{A}} \otimes {}^{t}h^{(\text{r})}. \tag{2.57}$$

This results in the formula (2.54).

Once a candidate for the set of matrices $\{\widetilde{\mathbf{A}}, \mathbf{A}_{\text{c}}, |\widetilde{V}\rangle, \langle\widetilde{W}|\}$ is obtained from (2.54), we must check whether the candidate satisfies equations (2.50)–(2.52). If these equations hold, it means that the obtained set of matrices is valid for any L.

So far we have not succeeded in extending the formula (2.54) to the case $N \neq M$. This is an open question.

3. Examples

In this section, the application of our methods in section 2 is illustrated by performing explicit calculations for three models:

1. the asymmetric simple exclusion process (ASEP) [6, 16, 20–25] (section 3.1),
2. the model studied in [26] (section 3.2),
3. a hybrid of model (1) and model (2) (section 3.3).

For the first two models, it is already known that there exist MPSSs for special values of the model parameters. We will see that the conditions and the matrices can be reproduced from our methods. For the third model, we apply the same methods and find an MPSS, which was unknown.

3.1. The asymmetric simple exclusion process (ASEP)

In this subsection, we treat the asymmetric simple exclusion process (ASEP). The model is defined on the one-dimensional lattice whose size is L. Each site can take two states: a site is either empty or occupied by a particle. The time of the model is a continuous one (i.e. the random sequential update). A particle in the bulk hops to the left (resp. right) neighbour site with a rate p_{L} (resp. p_{R}) if the left (resp. right) neighbour site is empty. A particle at the rightmost site is removed with a rate β if the site is occupied by a particle. At the leftmost site, a particle is injected with a rate α if the site is empty.

The total Hamiltonian H of this model has the form in (2.46)–(2.49). For the case of the ASEP, $h^{(\ell)}$ of (2.48) and $h^{(\text{r})}$ of (2.49), which express the left and right boundary condition respectively, are

$$h^{(\ell)} := \begin{bmatrix} \alpha & 0 \\ -\alpha & 0 \end{bmatrix} \qquad \text{and} \qquad h^{(\text{r})} := \begin{bmatrix} 0 & -\beta \\ 0 & \beta \end{bmatrix}, \tag{3.1}$$

in a basis of states whose order is $(|\emptyset\rangle, |\text{A}\rangle)$ where $\emptyset$ and A represent an empty site and a site occupied by a particle, respectively. The interaction Hamiltonian h_{int} of (2.47) is defined as

$$h_{\text{int}} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.2}$$

in a basis of states whose order is $(|\emptyset\emptyset\rangle, |\emptyset\text{A}\rangle, |\text{A}\emptyset\rangle, |\text{AA}\rangle)$. In (3.2), we have introduced the parameter $q := p_{\text{L}}/p_{\text{R}}$ and set $p_{\text{R}} = 1$. In the following, we assume that $\alpha > 0$, $\beta > 0$, $q > 0$.

3.1.1. Determination of the necessary condition for the existence of an M-dimensional MPSS—a case of the ASEP. Let us find a necessary condition for the existence of the $M(=1,2)$-dimensional set of matrices $\{E, D, |V\rangle, \langle W|\}$. We perform the following calculations according to the procedure in section 2.1. First, we obtain $P^{2,2}$ (for its definition, see equation (2.9)) by solving equation (2.2) for the $(L=)4$-site system. Next we perform a set of elementary transformations so that the resultant matrix has the form in (2.21). For the present case, the resultant matrix is

$$\begin{cases} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{q+\beta}{\beta} \\ 0 & 0 & 1 & \beta^{-1} \\ 0 & 0 & 0 & 0 \end{bmatrix} & (f_{\mathrm{ASEP1}} \neq 0 \text{ and } f_{\mathrm{ASEP2}} \neq 0) \\ \begin{bmatrix} 1 & -\frac{-1+\beta+q}{\beta} & -\frac{-1+\beta+q}{\beta} & \frac{(-1+\beta+q)^2}{\beta^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & (f_{\mathrm{ASEP1}} = 0) \\ \begin{bmatrix} 1 & 0 & -\frac{(-1+\beta+q)^2 q}{\beta(q+\beta)} & -\frac{(-1+\beta+q)^2 q}{(q+\beta)\beta^2} \\ 0 & 1 & -\frac{-2q-\beta+q^2+\beta q}{q+\beta} & -\frac{(-1+\beta+q)(2q+\beta)}{\beta(q+\beta)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & (f_{\mathrm{ASEP1}} \neq 0 \text{ and } f_{\mathrm{ASEP2}} = 0) \end{cases} \tag{3.3}$$

where $f_{\mathrm{ASEP1}} := q + \alpha + \beta - 1$ and $f_{\mathrm{ASEP2}} := q^2 + q(\alpha + \beta - 1) + \alpha\beta$. From (3.3), we can see that the rank of $P^{2,2}$ is 1 when $f_{\mathrm{ASEP1}} = 0$. So $f_{\mathrm{ASEP1}} = 0$ is the necessary condition for the existence of an $M(=1)$-dimensional MPSS. According to [20–23], this condition is the true one. We can also see that the rank of $P^{2,2}$ is 2 when $f_{\mathrm{ASEP2}} = 0$ and $f_{\mathrm{ASEP1}} \neq 0$ from (3.3). So this condition is the necessary condition for the existence of an $M(=2)$-dimensional MPSS. This is also true according to [20–23]. In the next section 3.1.2, we try to find a two-dimensional set of matrices by using $f_{\mathrm{ASEP2}} = 0$. That is, we use

$$\alpha = g_{\mathrm{A}}(q, \beta), \qquad \text{where} \quad g_{\mathrm{A}}(q, \beta) := \frac{-q(q-1+\beta)}{q+\beta}. \tag{3.4}$$

Here we describe the rank deficiency which happens for the ASEP due to the existence of the algebraic relations, such as

$$DE - qED \propto (E + D) \tag{3.5}$$

and

$$D|V\rangle \propto |V\rangle. \tag{3.6}$$

These equations can be shown to be satisfied using (3.2), (2.50), (2.52), (3.1) and (3.12). Because of (3.5) and (3.6), the rank of $P^{2,2}$ is at most 3 (cf (3.3)). More generally, the rank of $\langle W|\binom{E}{D}^{\otimes k}(E\,D)^{\otimes 2}|V\rangle$ $(k = 2, 3, 4, \ldots)$ is also at most 3. It should be noted that, generally, not only the existence of an MPSS but also an algebraic relation between matrices in the MPSS could cause a rank deficiency.

3.1.2. Construction of the $(M=)2$-dimensional set of matrices—a case of the ASEP. Let us find the $(M=)2$-dimensional representation of the MPSS of the ASEP. In the following, we use expressions for which we have eliminated α by using (3.4).

Firstly, we calculate $\{\widetilde{E}, \widetilde{D}\}$. We solve equation (2.2) for $L = 2$ and transform the solution into the matrix form $P^{1,1}$:

$$Z_2 P^{1,1} = \begin{bmatrix} 1 & \frac{g_{\rm A}(q,\beta)}{\beta} \\ \frac{(q+q\beta+\beta^2)g_{\rm A}(q,\beta)}{\beta(q+\beta)} & \frac{g_{\rm A}(q,\beta)^2}{\beta^2} \end{bmatrix}.$$

From this equation, we define $\langle W_1|$ and $\langle W_2|$ by

$$Z_2 P^{1,1} =: \begin{pmatrix} \langle W_1| \\ \langle W_2| \end{pmatrix}. \tag{3.7}$$

We solve equation (2.2) also for $L = 3$ and transform the solution into the matrix form $P^{1,2}$:

$$Z_3 P^{1,2} = \begin{bmatrix} 1 & \frac{g_{\rm A}(q,\beta)}{\beta} & \frac{(q+q\beta+\beta^2)g_{\rm A}(q,\beta)}{\beta(q+\beta)} & \frac{g_{\rm A}(q,\beta)^2}{\beta^2} \\ \frac{(q^2+2q^2\beta+3q\beta^2+\beta^3)g_{\rm A}(q,\beta)}{(q+\beta)^2\beta} & \frac{(q+q\beta+\beta^2)g_{\rm A}(q,\beta)^2}{(q+\beta)\beta^2} & \frac{(q^2\beta+2q\beta^2+q+\beta^3)g_{\rm A}(q,\beta)^2}{(q+\beta)\beta^2} & \frac{g_{\rm A}(q,\beta)^3}{\beta^3} \end{bmatrix}.$$

From (2.36) and (2.37), we can obtain $\widetilde{E}$ and $\widetilde{D}$:

$$\widetilde{E} = \begin{bmatrix} \frac{2q+\beta}{q+\beta} & \frac{g_{\rm A}(q,\beta)}{\beta} \\ \frac{\beta}{q-1+\beta} & 0 \end{bmatrix} \qquad \text{and} \qquad \widetilde{D} = \frac{g_{\rm A}(q,\beta)}{\beta} \begin{bmatrix} q+\beta & 0 \\ \beta & 1 \end{bmatrix}.$$

Secondly, we calculate $\{\langle\widetilde{W}|, |\widetilde{V}\rangle\}$. For $|\widetilde{V}\rangle$, we can use

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \widetilde{E}|\widetilde{V}\rangle, \tag{3.8}$$

or

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \widetilde{D}|\widetilde{V}\rangle \tag{3.9}$$

which are the left and right half of (2.29), respectively. If $\det(\widetilde{E}) = q/(q+\beta) \neq 0$, we can obtain $|\widetilde{V}\rangle$ by $(\widetilde{E})^{-1}\binom{1}{0}$. Or, if $\det(\widetilde{D}) = q^2(q-1+\beta)^2/(\beta^2(q+\beta)) \neq 0$, we can obtain $|\widetilde{V}\rangle$ by $(\widetilde{D})^{-1}\binom{0}{1}$: $|\widetilde{V}\rangle = (\widetilde{E})^{-1}\binom{1}{0} = (\widetilde{D})^{-1}\binom{0}{1} = \begin{pmatrix} 0 \\ \frac{\beta}{g_{\rm A}(q,\beta)} \end{pmatrix}$. Under the same condition, we can get $\langle\widetilde{W}|$ by

$$\langle W_1| = \langle\widetilde{W}|\widetilde{E}, \tag{3.10}$$

or

$$\langle W_2| = \langle\widetilde{W}|\widetilde{D}, \tag{3.11}$$

which can be shown from (3.7) and (2.30). Thus $\langle\widetilde{W}| = \langle W_1|(\widetilde{E})^{-1} = \langle W_2|(\widetilde{D})^{-1} = \left(1, \frac{g_{\rm A}(q,\beta)}{\beta}\right)$.

At this point, we should check that $\vec{P}^{\rm MPF}$ (see (2.4)) calculated by our expression $\{\widetilde{E}, \widetilde{D}, \langle\widetilde{W}|, |\widetilde{V}\rangle\}$ is also the solution of (2.2) for any size L. For this check, it is adequate that there exist $E_{\rm c}$ and $D_{\rm c}$ which satisfy equations (2.50)–(2.52). Using the above-mentioned result, we obtain a candidate for $\{E_{\rm c}, D_{\rm c}\}$ by (2.54)

$$-E_{\rm c} = D_{\rm c} = g_{\rm A}(q,\beta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.12}$$

It is easy to show this $\{E_{\rm c}, D_{\rm c}\}$ satisfies equations (2.50)–(2.52). Therefore, it is not only a candidate but also the true $\{E_{\rm c}, D_{\rm c}\}$.

We comment on the relation between the known set of matrices and ours. The known two-dimensional matrices, $\{E, D\}$ [20–23], which we denote by $\{E_{\rm A}, D_{\rm A}\}$, are given by

$$E_{\mathrm{A}} := \begin{bmatrix} \frac{1-q}{g_{\mathrm{A}}(q,\beta)} & 0 \\ 1 & (1-q)\left(1+\frac{q}{g_{\mathrm{A}}(q,\beta)}\right) \end{bmatrix} \quad \text{and} \quad D_{\mathrm{A}} := \begin{bmatrix} \frac{1-q}{\beta} & (1-q)\left(1-\frac{1}{q}\right) \\ 0 & (1-q)\left(1+\frac{q}{\beta}\right) \end{bmatrix}.$$

The relation between $\{E_{\mathrm{A}}, D_{\mathrm{A}}\}$ and $\{\widetilde{E}, \widetilde{D}\}$ is

$$S_{\mathrm{A}} \begin{pmatrix} \widetilde{E} \\ \widetilde{D} \end{pmatrix} (S_{\mathrm{A}})^{-1} \frac{(1-q)}{g_{\mathrm{A}}(q,\beta)} = \begin{pmatrix} E_{\mathrm{A}} \\ D_{\mathrm{A}} \end{pmatrix}, \qquad \text{where} \quad S_{\mathrm{A}} := c \begin{bmatrix} \frac{1-q}{g_{\mathrm{A}}(q,\beta)} & \frac{1-q}{\beta} \\ 1 & 0 \end{bmatrix} \tag{3.13}$$

and c is a free parameter. This S_{A} exists when $\det S_{\mathrm{A}} = -\frac{c^2(1-q)}{\beta} \neq 0$ is satisfied. Moreover, our result (3.12) corresponds to the result in [20–23]. This is because $\left\{E_{\mathrm{c}}^{\mathrm{A}}, D_{\mathrm{c}}^{\mathrm{A}}\right\}$ defined as $\begin{pmatrix} E_{\mathrm{c}}^{\mathrm{A}} \\ D_{\mathrm{c}}^{\mathrm{A}} \end{pmatrix} := S_{\mathrm{A}} \begin{pmatrix} E_{\mathrm{c}} \\ D_{\mathrm{c}} \end{pmatrix} (S_{\mathrm{A}})^{-1} \frac{(1-q)}{g_{\mathrm{A}}(q,\beta)}$ are of type of (a scalar) × (the two-dimensional identity matrix).

Furthermore, we have checked that there exist relations like (3.13) between our ($M =$)3, 4-dimensional sets of matrices which are obtained by the method in section 2.2.1 and the sets of matrices in [21–23].

3.2. The model in [26]

The model treated in this subsection is the one studied in [26]. It is defined on the one-dimensional lattice which has L sites. Each site can be either empty or occupied by a particle. The time evolution of the model is continuous (i.e. the random sequential update). Between nearest-neighbour sites of the chain, there stochastically occur three types of processes: diffusion, coagulation and decoagulation. The rate of each process is

$$\begin{aligned} &\emptyset + \mathrm{A} \xrightarrow{\text{rate:}q} \mathrm{A} + \emptyset \\ &\mathrm{A} + \emptyset \xrightarrow{\text{rate:}q^{-1}} \emptyset + \mathrm{A} \end{aligned} \qquad \text{(diffusion)}$$

$$\begin{aligned} &\mathrm{A} + \mathrm{A} \xrightarrow{\text{rate:}q} \mathrm{A} + \emptyset \\ &\mathrm{A} + \mathrm{A} \xrightarrow{\text{rate:}q^{-1}} \emptyset + \mathrm{A} \end{aligned} \qquad \text{(coagulation)}$$

$$\begin{aligned} &\emptyset + \mathrm{A} \xrightarrow{\text{rate:}\Delta q} \mathrm{A} + \mathrm{A} \\ &\mathrm{A} + \emptyset \xrightarrow{\text{rate:}\Delta q^{-1}} \mathrm{A} + \mathrm{A} \end{aligned} \qquad \text{(decoagulation)},$$

where ∅ and A represent an empty site and a site occupied by a particle, respectively. At the leftmost site, a particle is injected with a rate α if the site is empty and removed with a rate β if the site is occupied. This model has a reflective boundary condition at the rightmost site. In the following, we assume $q > 0, \alpha > 0, \beta > 0, \Delta > 0$, although there exists a (four-dimensional) MPSS, which we do not treat in this paper, also in a case with $\alpha = \beta = 0$ [19].

The total Hamiltonian H of this model takes the form in (2.46)–(2.49), where $h^{(\ell)}$ and $h^{(\mathrm{r})}$ are given by

$$h^{(\ell)} := \begin{bmatrix} \alpha & -\beta \\ -\alpha & \beta \end{bmatrix} \qquad \text{and} \qquad h^{(\mathrm{r})} := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.14}$$

in a basis of states whose order is $(|\emptyset\rangle, |\mathrm{A}\rangle)$, and the interaction term h_{int} is defined as

$$h_{\mathrm{int}} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (1+\Delta)q & -\frac{1}{q} & -\frac{1}{q} \\ 0 & -q & \frac{1+\Delta}{q} & -q \\ 0 & -\Delta q & -\frac{\Delta}{q} & q+\frac{1}{q} \end{bmatrix} \tag{3.15}$$

in a basis of states whose order is $(|\emptyset\emptyset\rangle, |\emptyset\mathrm{A}\rangle, |\mathrm{A}\emptyset\rangle, |\mathrm{AA}\rangle)$. It is noted that the overall signs of (3.14) and (3.15) are opposite to the ones in [26].

3.2.1. Determination of the necessary condition for the existence of an M-dimensional MPSS—a case of the model in [26]. Let us find a necessary condition for the existence of the $(M=)2$-dimensional set of matrices $\{E, D, |V\rangle, \langle W|\}$. According to the procedure in section 2.1, we solve equation (2.2) for $L = 4$ and obtain the four-dimensional square matrix $P^{2,2}$ (for its definition, see equation (2.9)). We perform a set of elementary transformations so that the resultant matrix has the form in (2.21). The resultant matrix is

$$\begin{cases} I_4 & (f_{\mathrm{J}} \neq 0) \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q^2 & \Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & (f_{\mathrm{J}} = 0) \end{cases} \tag{3.16}$$

where I_4 is the four-dimensional identity matrix and $f_{\mathrm{J}} := \Delta\beta q + \Delta - q\alpha - q^2\Delta$. In the process of transformation to obtain (3.16), two 'big' polynomials, each of which has more than 493 terms, appear chiefly in factors of numerators of the diagonal elements. It seems very unlikely that these polynomials contain important information about the existence of an MPSS. Hence we assume that these polynomials cannot vanish, so that we can divide by them. From (3.16), we can tell that the rank of $P^{2,2}$ changes from 4 to 2 when $f_{\mathrm{J}} = 0$. This condition, $f_{\mathrm{J}} = 0$, can be rewritten as

$$\alpha = g_{\mathrm{J}}(q, \beta, \Delta), \qquad \text{where} \quad g_{\mathrm{J}}(q, \beta, \Delta) := (1/q - q + \beta)\Delta. \tag{3.17}$$

In the next section 3.2.2, we assume this condition (3.17) holds.

3.2.2. Construction of the $(M=)2$-dimensional set of matrices—a case of the model in [26]. Let us find the $(M=)2$-dimensional representation of the MPSS of the model. Because the calculations we have to perform here are quite similar to those in section 3.1.2, we mainly focus on summarizing the results, for which we have eliminated α using (3.17). The set of matrices of this model is

$$\widetilde{E} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{q^2} \end{bmatrix} \qquad \text{and} \qquad \widetilde{D} = \begin{bmatrix} 0 & 0 \\ 1 & \frac{\Delta}{q^2} \end{bmatrix} \qquad \left(\text{if} \quad \Delta \neq \frac{g_{\mathrm{J}}(q, \beta, \Delta)}{\beta}\right)$$

$$|\widetilde{V}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \langle\widetilde{W}| = \begin{bmatrix} 1 & \frac{g_{\mathrm{J}}(q, \beta, \Delta)}{\beta} \end{bmatrix}.$$

The candidates for E_{c} and D_{c} which can be obtained from (2.54) are

$$E_{\mathrm{c}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\Delta(q^2-1)}{q^3} \end{bmatrix} \qquad \text{and} \qquad D_{\mathrm{c}} = -E_{\mathrm{c}}.$$

It is easy to check that these also satisfy equations (2.50)–(2.52). Therefore, they are the true E_{c} and D_{c}.

3.3. A hybrid model

In this subsection, we treat a model similar to the one in section 3.2. The only differences are boundary conditions. The boundary conditions of the model treated here are the same as those of the model in section 3.1. Therefore the total Hamiltonian H of this model is defined by

(2.46)–(2.49) where $h^{(\ell)}$ and $h^{(r)}$ are the same Hamiltonians as those defined in (3.1) and the interaction Hamiltonian h_{int} is the same as that of the model in [26] (except an overall sign) and is given by (3.15). In the following calculations, we assume $q > 0, \alpha > 0, \beta > 0, \Delta > 0$.

3.3.1. Determination of the necessary condition for the existence of an M-dimensional MPSS—a case of a hybrid model. Let us find a necessary condition for the existence of the $(M=)2$-dimensional set of matrices $\{E, D, |V\rangle, \langle W|\}$. We perform the following calculations as in section 3.1.1 according to the procedure in section 2.1. We solve equation (2.2) for $L = 4$ and obtain the four-dimensional square matrix $P^{2,2}$. The resultant matrix after a set of elementary transformations is

$$\begin{cases} I_4 & (f_{\text{hy}} \neq 0) \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q(q+\beta) & \Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & (f_{\text{hy}} = 0) \end{cases} \tag{3.18}$$

where I_4 is the four-dimensional identity matrix and $f_{\text{hy}} := -\Delta + q^2\Delta + q\alpha$. As in section 3.2.1, in the process of transforming to (3.18), two 'big' polynomials, each of which has more than 501 terms, appear chiefly in factors of numerators of the diagonal elements. We again assume that these polynomials cannot be zero, so that we can divide by them. From (3.18), we can tell that the rank deficiency occurs (the change of the rank is 4 to 2) when $f_{\text{hy}} = 0$. We regard the condition, $f_{\text{hy}} = 0$, as a necessary condition for the existence of an $M(=2)$-dimensional MPSS. For use in the next subsection 3.3.2, we transform $f_{\text{hy}} = 0$ into the following form

$$\alpha = g_{\text{hy}}(q, \Delta) \qquad \text{where} \quad g_{\text{hy}}(q, \Delta) := \frac{\Delta(1-q^2)}{q}. \tag{3.19}$$

3.3.2. Construction of the $(M=)2$-dimensional set of matrices—a case of a hybrid model. Let us find the $(M=)2$-dimensional representation of the MPSS of the hybrid model. Because calculations we have to perform here are quite similar to those in section 3.1.2, we mainly focus on summarizing the results, for which we have eliminated α by using (3.19). The set of matrices of this model is

$$\widetilde{E} = \begin{bmatrix} q^2 & 0 \\ \frac{\beta q}{\Delta} & 1 \end{bmatrix} \quad \widetilde{D} = \begin{bmatrix} 0 & 0 \\ q(q+\beta) & \Delta \end{bmatrix}$$

(if $\Delta \neq q(q+\beta)g_{\text{hy}}(q, \Delta)/\beta$)

$$|\widetilde{V}\rangle = \begin{bmatrix} 1/q^2 \\ -\frac{\beta}{\Delta q} \end{bmatrix} \quad \langle \widetilde{W}| = \begin{bmatrix} 1 & \frac{g_{\text{hy}}(q, \Delta)}{\beta} \end{bmatrix}.$$

And candidates for E_c and D_c, which are obtained from (2.54) are

$$-E_c = D_c = \begin{bmatrix} 0 & 0 \\ \beta & g_{\text{hy}}(q, \Delta) \end{bmatrix}. \tag{3.20}$$

We have checked that these candidates for E_c and D_c satisfy equations (2.50)–(2.52), so that (3.20) represents the true E_c and D_c.

4. Summary

In this paper, we have given a systematic way to find and construct exact finite-dimensional matrix product stationary states for one-dimensional stochastic models which have N states per site. More precisely, we have explained (section 2):

1. a systematic way to search necessary conditions for the existence of an *exact* M-dimensional matrix-product stationary state (MPSS) (section 2.1),
2. a systematic way by which the exact $M(\geqslant N)$-dimensional representation of the matrix product form (MPF) can be constructed from the stationary states of the small L if the necessary condition obtained in the above-mentioned step 1 is also a sufficient condition for the existence of an M-dimensional MPSS (section 2.2),
3. a systematic way to check the validity of the obtained MPSS for an arbitrary system size in the case $N = M$ (section 2.2.2).

After giving the general ideas, we have presented three examples to which our methods can be applied (section 3):

1. the asymmetric simple exclusion process (ASEP) [6, 16, 20–25] (section 3.1)
2. the model in [26] (section 3.2)
3. a hybrid of 1 and 2 (section 3.3)

For the first two models, we have reproduced the known finite-dimensional MPSSs. For the hybrid model, we have found a new MPSS by our method. This clearly shows the potential power of our method to find exact MPSSs for a large class of one-dimensional stochastic models.

Although we have treated only *exact* MPFs in this paper, we can also show how to construct *numerically* exact MPFs for more generic cases where calculations are so tedious that the computer algebra system (for example, Maple) is not useful (this method will be published elsewhere). It should also be noted that we have treated *uniform* MPFs in this paper, although *non-uniform* MPFs can be made numerically, for example, by the DMRG [11].

Finally we would like to mention possible extensions of our methods. The first step is to remove a restriction ($N = M$) on the applicability of our methods (section 2). This step helps to develop the applicability of our methods to various kinds of models (e.g. models with periodic boundary conditions; quantum spin chains) and also to develop the numerical renormalization (for example, the DMRG [11]) to stochastic models [27–34].

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Note added. After the completion of the manuscript, we noticed that our results for the hybrid model, i.e., the condition (3.19) and the two-dimensional set of matrices, had already been derived in [35] and [36] respectively. They were, however, new when they were discovered and hence we did not change the original presentation of this paper.

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